ON $\alpha$-OPERATORS

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Abstract

In this paper, we introduce the class $(\alpha)$ of operators acting on a complex Hilbert space $H$. We study some basic properties of operators in $(\alpha)$. We study the relation between the class $(\alpha)$ and some other classes of operators in $L(H)$. Finally, we study the sum and the product of two operators in $(\alpha)$.

1. Introduction

Let $H$ be a complex Hilbert space and let $L(H)$ be the algebra of all bounded linear operators acting on $H$. If $T \in L(H)$, then $T = A + Bi$ is its Cartesian decomposition, where $A$ and $B$ are self adjoint operators and $T^*$ is its adjoint. In Section 2 of this paper, we introduce the class of $\alpha$-operators, which we denote by $(\alpha)$ and we study some of its basic properties. In Section 3, we study the relation between the class $(\alpha)$ and some other classes of operators in $L(H)$. In Section 4, we study the sum and product of two operators in the class $(\alpha)$.

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2. The Class of $\alpha$ - Operators and its Basic Properties

In this section, we introduce $\alpha$ - operators acting on $H : T \in L(H)$ is called $\alpha$ - operator, if $T^3 = T^*$, and $\alpha$ - operators have some nice algebraic properties.

**Proposition 2.1.** If $T \in L(H)$ is an $\alpha$ - operator, then so are:

1. $T^*$;
2. $kT$ for $k = 0, 1, \text{ and } -1$;
3. $T^{-1}$ (if exists);
4. any $S \in L(H)$, which is unitarily equivalent to $T$;
5. the restriction $T / M$ of $T$ to any closed subspace $M$ of $H$, which reduces $T$.

**Proof.** (1) $(T^*)^3 = (T^3)^* = (T^*)^*$.

(2) $(kT)^3 = k^3T^3 = k^3T^*$.

Also, $(kT)^* = kT^*$.

Now, if $k = -1, 0 \text{ or } 1$, then $kT^* = k^3T^*$, which implies that $(kT)^* = (kT)^3$. Thus $kT$ is an $\alpha$ - operator.

(3) $(T^{-1})^3 = (T^3)^{-1} = (T^*)^{-1} = (T^{-1})^*$.

(4) Let $S \in L(H)$ be unitarily equivalent to $T$. Then there is a unitary operator $U \in L(H)$ such that $S = U^*TU$, which implies that $S^* = U^*T^*U$. Thus $T^* = US^*U^*$. Now, $S^3 = U^*T^3U = U^*T^*U = U^*US^*U^*U = S^*$. Thus $S$ is an $\alpha$ - operator.
\[(5) \ (T / M)^3 = T^3 / M = T^* / M = (T / M)^* \]. Thus \( T / M \) is an \( \alpha \)-operator.

The following example shows that unitarily equivalence in Proposition 2.1(4) cannot be replaced by similarity:

**Example 2.1.** Consider the operators \( S = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, X = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \) and 
\[
T = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix}
\]
acting on \( \mathbb{R}^2 \). Then by direct calculations, one can show that \( S^3 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = S^* \). Thus \( S \) is an \( \alpha \)-operator. Also, by direct calculation, one can show that \( X^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \) and \( T = X^{-1} SX \), which means that \( T \) is similar to \( S \). Now, direct calculations show that 
\[
T^3 = \begin{pmatrix} 2 & 1 \\ -2 & -1 \end{pmatrix} \neq \begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} = T^*.
\]
Thus \( T \) is not an \( \alpha \)-operator.

**Proposition 2.2.** The zero operator, \( 0 \), and the identity operator, \( I \), are \( \alpha \)-operators.

**Proof.** The proof is direct.

**Proposition 2.3.** The class \((\alpha)\) is closed in the strong operator topology.

**Proof.** Let \( \{T_n\} \) be a sequence of operators in \((\alpha)\) that converges strongly to \( T \in L(H) \). Since the product of operators is sequentially continuous in the strong operator topology ([1], p. 62), we conclude that \( \{T_n^3\} \) converges strongly to \( T^3 \). Now, since \( \{T_n\} \) converges strongly to \( T \), we have \( \|T_n x - Tx\| \to 0 \) as \( n \to \infty \) for each \( x \in H \). Thus

\[
\|T_n^* x - T^* x\| = \|(T_n^* - T^*) x\| \leq \|T_n - T\|^* \|x\| = \|T_n - T\| \|x\| \to 0
\]
as \( n \to \infty \). Thus \( \{T_n^*\} \) converges strongly to \( T^* \), which implies that
\( \{T_n^3\} \) converges strongly to \( T^* \). Since the limit is unique, we have \( T^3 = T^* \). Thus \( T \in (\alpha) \), which implies that \( (\alpha) \) is closed in the strong operator topology.

### 3. The Relation Between \( \alpha \)-Operators and Some Other Classes of Operators in \( L(H) \)

In this section, we study the relation between the class \( (\alpha) \) and some other classes of operators in \( L(H) \).

**Proposition 3.1.** If \( T \in L(H) \) is an \( \alpha \)-operator, then \( T \) is normal.

**Proof.** Since \( T \) is \( \alpha \)-operator, \( T^3 = T^* \). Multiplying both sides of the last equation on the left and then on the right by \( T \), we get \( T^4 = T^*T = TT^* \). Thus \( T \) is normal.

The converse of Proposition 3.1 is not in general true. This can be shown in the following example:

**Example 3.1.** Consider the following operator \( T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \) acting on the two dimensional Hilbert space \( \mathbb{R}^2 \). Then \( T \) is Hermitian, thus normal. Now, by direct calculations, one can show that \( T^3 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = T^* \). Thus \( T \) is not an \( \alpha \)-operator.

**Remark 3.1.** Example 3.1 serves as an example of a Hermitian operator, which is not \( \alpha \)-operator.

**Example 3.2.** Consider the operator \( T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \) acting on the two dimensional Hilbert space \( \mathbb{R}^2 \). Then it is clear that \( T \) is not Hermitian.
However, by direct calculations, one can show that $T^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = T^*.$

Thus $T$ is an $\alpha$-operator. From the last example and Example 3.1, we conclude that the two classes of Hermitian operators and $\alpha$-operator are independent.

The following is a condition on an $\alpha$-operator to be Hermitian.

**Proposition 3.2.** If $T \in L(H)$ is an $\alpha$-operator such that $T$ is an idempotent, then $T$ is Hermitian.

**Proof.** The proof is direct.

**Proposition 3.3.** If $T \in L(H)$ is a projection, then it is an $\alpha$-operator.

**Proof.** Since $T$ is a projection, $T^2 = T$ and $T^2 = T^*$. Now, $T^3 = T^2T = TT = T^2 = T^*$. Thus $T \in (\alpha)$. The converse of Proposition 3.2 is not in general true. As an example of an $\alpha$-operator, which is not a projection consider the operator $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ acting on $R^2$.

The following is a condition that makes an operator $T \in (\alpha)$ a projection.

**Proposition 3.4.** If both $T$, $I - T \in (\alpha)$, then $T$ is a projection.

**Proof.**

\[(I - T)^3 = I - 3T + 3T^2 - T^3, \quad \text{(i)}\]
\[(I - T)^* = I - T^*. \quad \text{(ii)}\]

Since $I - T \in (\alpha)$, the left hand sides of Equations (i) and (ii) above are equal. Thus $I - 3T + 3T^2 - T^3 = I - T^*$. Since $T \in (\alpha)$, we have
$T = T^2$. Now, multiplying the last equation on both sides by $T$, we get $T^3 = T^2$, which implies that $T = T^*$. Thus $T$ is a projection.

In [4], Kamei introduced the class of skew normal operators acting on $H$. If $T = A + Bi \in L(H)$, then $T$ is called skew-normal, if $AB = -BA$. It follows immediately from the definition that $T$ is skew normal, if and only if $T^2$ is Hermitian.

**Proposition 3.5.** If $T \in L(H)$ is skew normal, then it is not necessary that $T$ is an $\alpha$-operator.

**Proof.** We prove the result by an example.

**Example 3.3.** Let $T = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$ be an operator acting on $R^2$. Then

$T^2 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$. Thus $T$ is skew normal. However, one can easily show that

$T^3 = \begin{pmatrix} 1 & 0 \\ 0 & -8 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} = T^*$. Thus $T$ is not an $\alpha$-operator.

**Proposition 3.6.** If $T \in L(H)$ is skew normal and isometry, then $T$ is an $\alpha$-operator.

**Proof.** Since $T$ is an isometry, $T^*T = I$. Multiplying the last equation on the left by $T^*$, we get $T^*T^*T = T^*$, which implies that $(T^*)^2 T = T^*$. Thus, we have $(T^2)^* T = T^*$. Since $T$ is skew normal, $T^2$ is Hermitian, so the last equation becomes $T^2T = T^3 = T^*$. Thus $T$ is an $\alpha$-operator.

In [2], the author introduced the class of subprojections acting on $H$: $T \in L(H)$ is called a subprojection, if $T^2 = T^*$.

The following is an example of an operator in $(\alpha)$, which is not a subprojection.
Example 3.4. Consider the operator $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ acting on $\mathbb{R}^2$.

Then it can be easily shown that $T^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \neq T^* = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Thus $T$ is not a subprojection. However, one can easily show that $T^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = T^*$. Thus $T$ is an $\alpha$-operator.

The following is an example of a subprojection operator, which is not in (a).

Example 3.5. Consider the operator $T = \begin{pmatrix} -1 & \sqrt{3} \\ \frac{2}{\sqrt{3}} & 2 \\ \frac{2}{\sqrt{3}} & -1 \\ 2 & 2 \end{pmatrix}$ acting on $\mathbb{R}^2$.

Then it can be easily shown that $T^2 = \begin{pmatrix} -1 & -\sqrt{3} \\ \frac{2}{\sqrt{3}} & 2 \\ \frac{2}{\sqrt{3}} & -1 \\ 2 & 2 \end{pmatrix} = T^* = \begin{pmatrix} \frac{1}{2} & -\sqrt{3} \\ \frac{2}{\sqrt{3}} & 2 \\ \frac{2}{\sqrt{3}} & -1 \\ 2 & 2 \end{pmatrix}$.

Thus $T$ is a subprojection. However, one can show that $T^3 = I \neq T^*$. Thus $T$ is not an $\alpha$-operator. From the last two examples, we conclude that the class (a) and the class of all subprojection operators are independent. In the following, we give a characterization of a projection in $L(H)$.

Proposition 3.7. $T \in L(H)$ is a projection, if and only if $T$ is both a subprojection and an $\alpha$-operator.

Proof. Let $T$ be a projection. Then $T^2 = T^* = T$, which means that $T$ is a subprojection. Also $T^2 = T$ implies that $T^3 = T^2 = T^*$. Thus $T$ is an $\alpha$-operator.

Now, suppose that $T$ is both a subprojection and an $\alpha$-operator. Then $T^3 = T^2$, which implies that $T^4 = T^3 = T^* = T^2$. Thus $T^4 = T^2$. 

which implies that $T^2T^2 = T^2$. Thus $T^*T^* = T^2$, which implies that $T^*T^2 = (T^2)^* = T^2$. Thus $T^2$ is Hermitian, which implies that $T$ is skew-normal. Now by ([2], Proposition 4.2, p. 235), $T$ is a projection.

In [3], the author introduced the class of almost subprojection operators: $T \in L(H)$ is almost subprojection, if $T^2$ is a subprojection. It follows immediately that $T$ is almost subprojection, if $T^4 = T^{*2}$.

In the following, we give two examples to show that the class of almost subprojection operators and the class $(\alpha)$ are independent.

**Example 3.6.** Consider the operator $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ acting on $R^2$. Then direct calculations show that $T^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq T^*$. Thus $T \notin (\alpha)$. However, by direct calculations, one can show that $T^4 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = T^{*2}$. Thus $T$ is an almost subprojection operator.

**Example 3.7.** Consider the operator $T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ acting on $R^2$. Then one can easily show that $T^3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = T^*$. Thus $T$ is in $(\alpha)$. However, one can show by direct calculations that $T^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = T^*$. Thus $T$ is not an almost subprojection.

In the following, we give an example of an operator in $(\alpha)$, which is not an isometry.

**Example 3.8.** Consider the operator $T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ acting on $R^2$. Then it is easy to show that $T \in (\alpha)$. However, by direct calculations, one can show that $T^*T = T \neq I$. Thus $T$ is not an isometry.
In the following, we give an example of an isometry, which is not an $\alpha$-operator.

**Example 3.9.** Consider the operator 
$$
T = \begin{pmatrix}
-1 & \sqrt{3} \\
2 & 2 \\
-\sqrt{3} & -1 \\
2 & 2
\end{pmatrix}
$$
acting on $\mathbb{R}^2$.

Then by direct calculations, one can show that $T^*T = I$. Thus $T$ is an isometry.

However, by direct calculations, one can show that $T^3 = I \neq T^*$. Thus $T$ is not an $\alpha$-operator. From the last two examples, we conclude that the class of all isometries and the class $(\alpha)$ are independent.

### 4. The Sum and the Product of Two $\alpha$-Operators

In this section we study the sum and the product of two $\alpha$-operators.

**Proposition 4.1.** If $S, T \in (\alpha)$ such that $ST = TS$, then $ST \in (\alpha)$.

**Proof.** Since $ST = TS$, $(ST)^3 = T^3S^3 = T^*S^* = (ST)^*$. Thus $ST$ is $\alpha$-operator.

**Remark 4.1.** In Proposition 4.1, if $ST \neq TS$, then $ST$ may not be $\alpha$-operator. The following example shows this:

**Example 4.1.** Consider the operators 
$$
S = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix},
T = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
$$
acting on $\mathbb{R}^2$. Then each is an $\alpha$-operator. Now, by direct calculations, we have $ST = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Using direct calculation, we get $(ST)^3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = (ST)^*$. Thus $ST$ is not an $\alpha$-operator.
**Proposition 4.2.** The sum of two $\alpha$-operators in $L(H)$ may not be an $\alpha$-operator.

**Proof.** Let $T \in L(H)$ be an $\alpha$-operator. Then (by Proposition 1.1(2)) $T + T = 2T$ is not an $\alpha$-operator.

**Proposition 4.3.** The direct sum and the tensor product of two $\alpha$-operators are $\alpha$-operators.

**Proof.** Let $S, T$ be two $\alpha$-operators in $L(H)$ and let $x = x_1 \oplus x_2$ be an element in $H \oplus H$. Then

$$(S \oplus T)^3(x) = (S \oplus T)^3(x_1 \oplus x_2)$$

$$= S^3(x_1) \oplus T^3(x_2)$$

$$= S^*(x_1) \oplus T^*(x_2)$$

$$= (S^* \oplus T^*)(x_1 \oplus x_2)$$

$$= (S \oplus T)^\dagger(x).$$

Thus $(S \oplus T)^3 = (S \oplus T)^\dagger$, which implies that $S \oplus T$ is an $\alpha$-operator.

Also

$$(S \otimes T)^3(x) = (S \otimes T)^3(x_1 \oplus x_2)$$

$$= S^3(x_1) \otimes T^3(x_2)$$

$$= S^*(x_1) \otimes T^*(x_2)$$

$$= (S^* \otimes T^*)(x_1 \oplus x_2)$$

$$= (S \otimes T)^\dagger(x).$$

Thus $(S \otimes T)^3 = (S \otimes T)^\dagger$, which implies that $S \otimes T$ is an $\alpha$-operator.
References


